Finite difference method – Euler Method

Downloaded on 9/23/10 from Wikipedia, the free encyclopedia <http://en.wikipedia.org/wiki/Finite_difference_method>

Not to be confused with "finite difference method based on variation principle", the first name of finite element method.

In mathematics, **finite-difference methods** are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives.

Intuitive derivation

Finite-difference methods approximate the solutions to differential equations by replacing derivative expressions with approximately equivalent difference quotients. That is, because the first derivative of a function f is, by definition,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

then a reasonable approximation for that derivative would be to take

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

for some small value of h. In fact, this is the forward difference equation for the first derivative. Using this and similar formulae to replace derivative expressions in differential equations, one can approximate their solutions without the need for calculus.

Derivation from Taylor's polynomial

Assuming the function whose derivatives are to be approximated is properly-behaved, by Taylor's theorem,

$$f(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f^{(2)}(x_0)}{2!}h^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n + R_n(x),$$

where *n*! denotes the factorial of *n*, and $R_n(x)$ is a remainder term, denoting the difference between the Taylor polynomial of degree *n* and the original function. Again using the first derivative of the function *f* as an example, by Taylor's theorem,

$$f(x_0 + h) = f(x_0) + f'(x_0)h + R_1(x),$$

which, with some minor algebraic manipulation, is equivalent to

$$f'(a) = \frac{f(a+h) - f(a)}{h} + R_1(x)$$

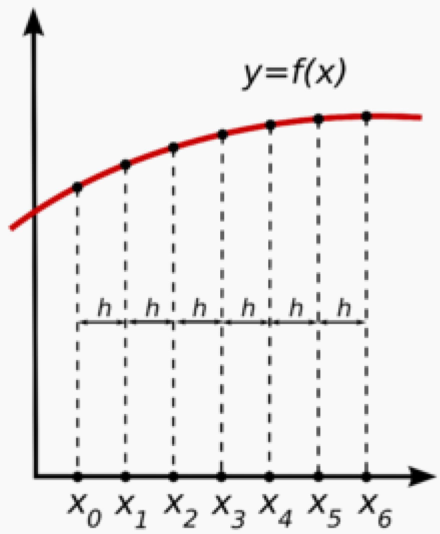
so that for $R_1(x)$ sufficiently small,

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$

Accuracy and order

See also: Finite difference coefficients

The error in a method's solution is defined as the difference between its approximation and the exact analytical solution. The two sources of error in finite difference methods are round-off error, the loss of precision due to computer rounding of decimal quantities, and truncation error or discretization error, the difference between the exact solution of the finite difference equation and the exact quantity assuming perfect arithmetic (that is, assuming no round-off).



The finite difference method relies on discretizing a function on a grid.

To use a finite difference method to attempt to solve (or, more generally, approximate the solution to) a problem, one must first discretize the problem's domain. This is usually done by dividing the domain into a uniform grid (see image to the right). Note that this means that finite-difference methods produce sets of discrete numerical approximations to the derivative, often in a "time-stepping" manner.

An expression of general interest is the local truncation error of a method. Typically expressed using Big-O notation, local truncation error refers to the error from a single application of a method. That is, it is the quantity $f'(x_i) - f'_i$ if $f'(x_i)$ refers to the exact value and f'_i to the numerical approximation. The remainder term of a Taylor polynomial is convenient for analyzing the local truncation error. Using the Lagrange form of the remainder from the Taylor polynomial for $f(x_0 + h)$, which is

$$R_n(x_0+h) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(h)^{n+1}$$, where $x_0 < \xi < x_0 + h$,

the dominant term of the local truncation error can be discovered. For example, again using the forward-difference formula for the first derivative, knowing that $f(x_i) = f(x_0 + ih)$,

$$f(x_0+ih) = f(x_0) + f'(x_0)ih + \frac{f''(\xi)}{2!}(ih)^2,$$

and with some algebraic manipulation, this leads to

$$\frac{f(x_0+ih)-f(x_0)}{ih} = f'(x_0) + \frac{f''(\xi)}{2!}ih,$$

and further noting that the quantity on the left is the approximation from the finite difference method and that the quantity on the right is the exact quantity of interest plus a remainder, clearly that remainder is the local truncation error. A final expression of this example and its order is:

$$\frac{f(x_0+ih)-f(x_0)}{ih} = f'(x_0) + O(h).$$

This means that, in this case, the local truncation error is proportional to the step size.

Example: ordinary differential equation

For example, consider the ordinary differential equation

$$u'(x) = 3u(x) + 2.$$

The Euler method for solving this equation uses the finite difference quotient

$$\frac{u(x+h)-u(x)}{h} \approx u'(x)$$

to approximate the differential equation by first substituting in for u'(x) and applying a little algebra to get

$$u(x+h) = u(x) + h(3u(x)+2).$$

The last equation is a finite-difference equation, and solving this equation gives an approximate solution to the differential equation.