Cramer's Rule: Short Examples

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* Solve $4x - 2y = 10$
$3x - 5y = 11$, or

$$\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}, \quad A\underline{X} = B \text{ form.}$$

Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$, $A_x = \begin{bmatrix} 10 & -2 \\ 11 & 5 \end{bmatrix}$, $A_y = \begin{bmatrix} 4 & 10 \\ 3 & 11 \end{bmatrix}$

Solutions: $x = \dfrac{|A_x|}{|A|}$ and $y = \dfrac{|A_y|}{|A|}$

$$x = \frac{\begin{vmatrix} 10 & -2 \\ 11 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix}} = \frac{(10)(5) - (11)(-2)}{(4)(-5) - (3)(-2)} = \frac{-28}{-14} = 2$$

$$y = \frac{\begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix}}{|A|} = \frac{(4)(11) - (3)(10)}{-14} = \frac{14}{-14} = -1$$

So the solution $(x, y) = (2, -1)$ for this system.

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* Now a 3x3. Solve
$-x + 2y - 3z = 1$
$2x + 0y + z = 0$ or,
$3x - 4y + 4z = 2$

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

in $A\underline{X} = B$ form.

$$\det A = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = -1\begin{vmatrix} 0 & 1 \\ -4 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 3 & -4 \end{vmatrix} = -1(4) - 2(5) - 3(-8) = 10$$

$$\det A_x = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ -4 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 0 & 0 \\ 2 & -4 \end{vmatrix} = 1(4) - 2(-2) - 3(0) = 8$$

$$\det A_y = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -1\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} = -1(-2) - 1(5) - 3(4) = -15$$

$$\det A_z = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -1\begin{vmatrix} 0 & 0 \\ -4 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 3 & -4 \end{vmatrix} = -1(0) - 2(4) + 1(-8) = -16$$

$x = \dfrac{|A_x|}{|A|} = \dfrac{8}{10} = \dfrac{4}{5}$, $y = \dfrac{|A_y|}{|A|} = \dfrac{-15}{10} = \dfrac{-3}{2}$, and $z = \dfrac{|A_z|}{|A|} = \dfrac{-16}{10} = \dfrac{-8}{5}$

Solution $(x, y, z) = \left(\dfrac{4}{5}, \dfrac{-3}{2}, \dfrac{-8}{5}\right)$

Use Cramer's Rule to solve these systems.

1. $x + 2y = 5$
$-x + y = 1$

Answer: $(1, 2)$

2. $3x + 4y = -2$
$5x + 3y = 4$

Answer: $(2, -2)$

3. $\begin{bmatrix} 20 & 8 \\ 12 & -24 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 21 \end{bmatrix}$

Answer: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ -\frac{1}{2} \end{bmatrix}$

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4. $4x - y - z = 1$
$2x + 2y + 3z = 10$
$5x - 2y - 2z = -1$

Answer: $(1, 1, 2)$

5. $3x + 4y + 4z = 11$
$4x - 4y + 6z = 11$
$6x - 6y - 0 = 3$

Answer: $(1, \frac{1}{2}, \frac{3}{2})$

6. $\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Answer: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

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